performed over the temperature range of 300–700°K for the TM_{010}, TM_{020}, and TM_{030} resonances. The values of the three resonant frequencies—2.2, 4.8, and 7.4 GHz-are shifted from the empty cavity values due to the presence of the sample. Based on the extended method, we find that the frequency shift data at all three frequencies yield a value of $\epsilon' \approx 5.64 \pm 0.05$. This result is also in very close agreement with previous measurements [8].

Equations (18) and (19) have been used to obtain ϵ'' from experimentally measured Q values with and without the sample. The results are plotted in Fig. 2 along with theoretical predictions [10] based on a model for microwave absorption that includes ionic conduction [8], defect-complex-dipole relaxation [8] and multi-phonon quasiresonance [11] processes. The results of the experimental measurements and the model predictions are in excellent, consistent agreement over a very large temperature range and for a significant range of frequencies.

IV. Conclusion

The measurement method described in this paper extends the validity of the cavity perturbation technique to larger samples and multiple cavity modes, provided that the difference between Q_w and Q_0 is still negligible in comparison with the measurement precision. This extended applicability is advantageous for determining microwave dielectric properties of many important ionic crystalline solids that are difficult to fabricate into very thin rods. As an illustration, the method has been successfully used to study the dielectric properties of NaCl crystals in the microwave frequency regime.

Acknowledgment

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Characterization of Microstrip Discontinuities Using Conformal Mapping and the Finite-Difference Time-Domain Method

Sunil Kapoor and John B. Schneider

Abstract— **Microstrip discontinuities are analyzed using Wheeler's waveguide model and the finite-difference time-domain (FDTD) method. Wheeler's model employs a conformal transformation to convert a microstrip into an enclosed waveguide structure. This permits the mapping of a discontinuous microstrip into a discontinuous, but enclosed, waveguide. The enclosed waveguide eliminates the difficulties usually associated with analysis of an open domain geometry. The FDTD technique is then used to calculate the scattering coefficients of the discontinuous waveguide. The features of this approach are: 1) it yields a smaller computational domain than that required to analyze the untransformed geometry; 2) it yields results over a band of frequencies; and 3) it is simple to implement. Results obtained using this scheme show good agreement with previously published results.**

I. Introduction

Many modern microwave and millimeter-wave integrated circuits guide the transmission of energy using microstrip lines (or asymmetrical striplines). The passive components in these circuits are often constructed from microstrip discontinuities. To analyze and synthesize microwave integrated circuits, it is essential to accurately model the frequency-dependent properties of these discontinuities. The frequency-dependent properties of microstrip lines, in the absence of discontinuities, can be obtained from simple empirical formulae that accurately describe the phase velocities and the characteristic impedances of the fundamental and higher-order modes [1]. In the presence of discontinuities, analysis becomes quite cumbersome and several solution techniques have been proposed. These techniques are based on any one of a number of methods including mode matching [2]–[7], finite-difference time-domain (FDTD) [8]–[11], method of moments (MoM) [12]–[16], finite element method (FEM) [17], [18], and the measured equation of invariance (MEI) [19].

All of the aforementioned techniques have inherent limitations. For example, solutions based on mode matching can become unwieldy for even slightly complicated geometries. MoM, FEM, and MEI solutions can be expensive when results are desired over a broad spectrum. Direct application of FDTD to these circuits can require the use of a large and/or fine mesh which, in turn, requires long computation times and large amounts of computer memory.

This paper presents a technique that is both simple to implement and computationally inexpensive. The technique works by converting the open microstrip structure into an enclosed waveguide using the conformal mapping technique described by Wheeler [20], [21] and then using the conventional FDTD technique to analyze the discontinuities. The conformal mapping reduces the problem to one with no stray fields which greatly reduces the size of the computational domain. Since this is a time-domain technique, results can be obtained over a band of frequencies via Fourier transforms. However, since the conformal mapping is only accurate at lower frequencies, the cost of using this simplified approach is that the

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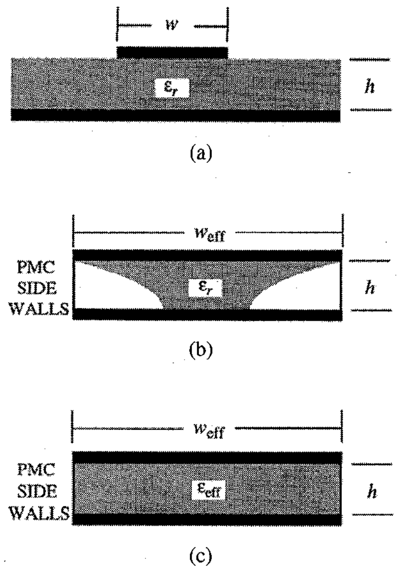

Fig. 1. (a) Cross section of microstrip line. (b) Inhomogeneous air-dielectric cross section of transformed line. (c) Homogeneous waveguide model of microstrip. The side walls of the transformed line are perfect magnetic conductors (PMC).

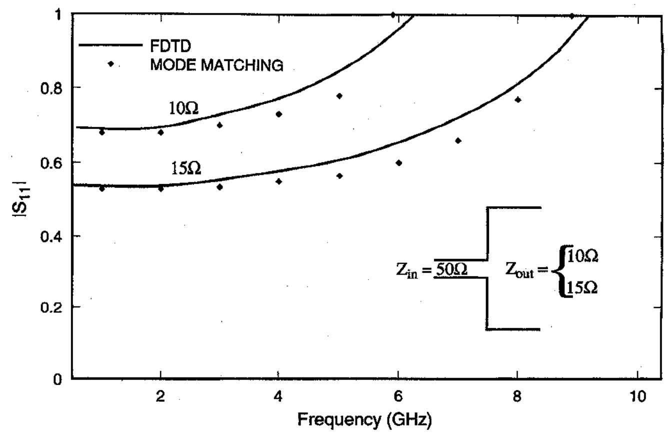

Fig. 2. Magnitude of the S_{11} scattering parameter as a function of frequency for a single-step discontinuity. At 1 GHz the input line has an impedance of 50 Ω while the output line has an impedance of either 10 or 15 Ω.

results are not as broad-banded as those obtained from a direct application of FDTD to the microstrip itself. An added cost is incurred by the inability of the model to account for losses due to radiation or the generation of surface waves [22]. Nevertheless, as will be shown in Section III, accurate results can be obtain for a broad class of structures over a moderately wide range of frequencies.

II. Implementation

Wheeler showed that a conformal mapping technique can be used to transform a microstrip line into an equivalent ideal parallel-plate waveguide if the fundamental EH mode on the microstrip line is quasi-TEM [20], [21] (which is correct at lower frequencies [1]). This waveguide has no stray fields since it is bounded by perfectly conducting magnetic side walls and perfectly conducting electric walls on top and bottom as shown in Fig. 1. The height h of the waveguide is identical to the thickness of the substrate material. The conformal mapping yields an inhomogeneous air-dielectric cross section as shown in Fig. 1(b). This inhomogeneous cross section can be approximated by a homogeneous cross section (Fig. 1(c)) with an effective dielectric constant $\varepsilon_{\rm eff}$ given by [21], [23]

$$\varepsilon_{\rm eff} = \left\{\sqrt{\varepsilon_r} + \frac{(\varepsilon_r - 1)\left[\ln\left(\frac{\pi}{4}\right)^2 + 1 - \varepsilon_r \ln\left(\frac{\pi e}{2}\right)\left(\frac{w}{2h} + 0.94\right)\right]}{2\varepsilon_r^{3/2}\left[\frac{w\pi}{2h} + \ln\left(2\pi e\left(\frac{w}{2h} + 0.94\right)\right)\right]}\right\}^2 \quad (1)$$

where ε_r is the permittivity of the substrate and w is the width of the microstrip. The effective width $w_{\rm eff}$ of the waveguide is given by [21], [23]

$$w_{\rm eff} = h\left[\frac{w}{h} + \frac{2}{\pi}\ln\left(2\pi e\left(\frac{w}{2h} + 0.92\right)\right)\right]. \quad (2)$$

Wheeler's waveguide model of the discontinuous microstrip line is not exact, but it does accurately describes the propagation of the fundamental mode which usually plays the most significant role in the propagation of energy. This model, when used in conjunction with other techniques such as mode-matching [2]–[7], has provided accurate descriptions of microstrip discontinuities, bends, and T-junctions. By using a temporal technique, such as FDTD, the waveguide model can be used to analyze the frequency-dependent transmission properties of the microstrip discontinuities. In this paper, FDTD is used in conjunction with the waveguide model as given by (1) and (2). However, the range of frequencies over which accurate results are obtained can, in theory, be extended by analyzing the stucture with the inhomogeneous cross section (Fig. 1(b)) rather than the one which incorporates an effective permittivity (Fig. 1(c)).

Here, the FDTD technique is used to synthesize and measure the propagation of fields within the waveguide model. The computational domain is relatively small and extends approximately one half of a wavelength to either side of the discontinuity (here the wavelength is assumed to be that of a 1 GHz signal). By using a pulse to illuminate the discontinuity and measuring both the transmitted and reflected energy, the scattering parameters of the device can be obtained over a range of frequencies. This is accomplished by obtaining the spectral description of the temporal signals (via fast Fourier transforms) and then normalizing the reflected and transmitted spectra by the incident spectrum.

The FDTD algorithm is inherently dispersive and anisotropic and thus introduces numerical artifacts. However, these artifacts are small if the spatial discretization is such that the principal spectral components are resolved with at least 10 points per wavelength [24]. This limits the error in the phase velocity to less than 1%, regardless of the direction of propagation within the grid. This accuracy in the phase velocity, coupled with the small computational domain, insures that the total phase error will be small. This approach has been used for the calculation of all results presented here. To insure stability, the discretization was chosen so that $c\delta t = \frac{1}{2}\Delta$ where δt is the temporal step size and Δ is the spatial step size (the step sizes in all three spatial directions were the same, i.e., $\Delta x = \Delta y = \Delta z = \Delta$). A TEM Gaussian pulse was used for illumination. The open ends of the waveguide structure were terminated using the third-order Liao absorbing boundary condition (ABC) [25]. Other effective techniques exist for the termination of these types of computational domains [26], [27].

III. Results

Fig. 2 shows the magnitude of the S_{11} scattering parameters obtained using the technique described above for two different single-step microstrip discontinuities. At 1 GHz, the TEM impedance for the input line is 50 Ω ($\varepsilon_r = 2.32$, $h = 1.57$ mm and $w = 4.68$ mm) while the impedance on the other side of the discontinuity is either 10 or 15 Ω ($w = 3.47$ cm or 2.2 cm). The results compare well with those previously obtained using a mode matching technique [23] (which also used the Wheeler transformation).

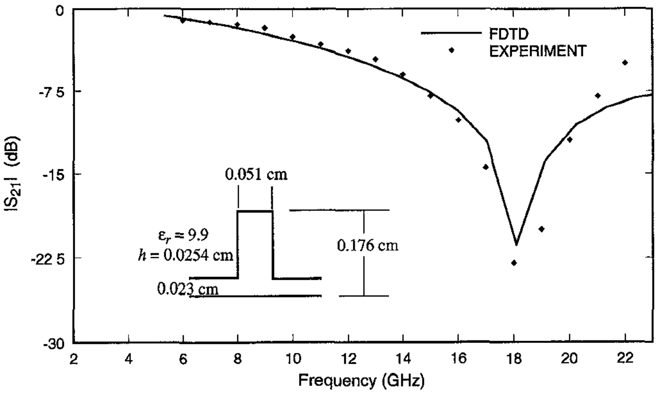

Fig. 3. Magnitude of the S_{21} scattering parameter as a function of frequency for a double-step discontinuity. Experimental values are from [28].

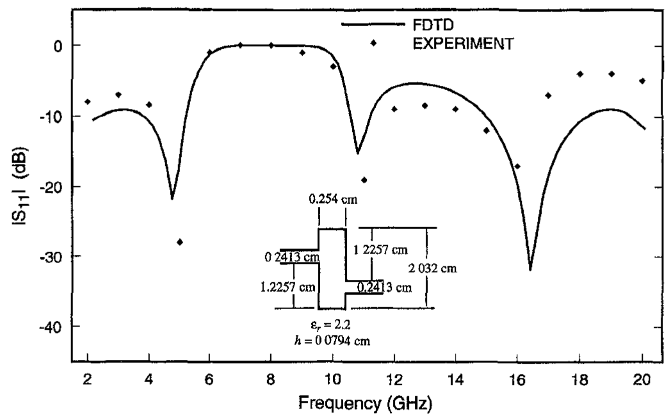

Fig. 4. Magnitude of the S_{11} scattering parameter as a function of frequency for a double-step discontinuity. Experimental values are from [10].

Figs. 3 and 4 show a comparison of results from the technique presented here to experimental results previously published by Giannini *et al.* [28] and Sheen *et al.* [10] for a double-step structure. The results show excellent agreement except at higher frequencies.

Sheen *et al.* [10] analyzed the structure shown in Fig. 4 using a three-dimensional FDTD method. In our implementation the code was relatively simple in that the spatial step size in the three coordinate directions was constant whereas in [10] different step sizes were used. The same temporal step $\Delta t = 0.441$ ps was used here as was used in [10]. Our simulation used 4096 time steps and took approximately one and a half hours to compute on an HP9000 workstation whereas the computation time for the full FDTD implementation of [10] was approximately eight hours on a DEC VAXstation 3500. No attempt was made to optimize our code and the computation time could be reduced by using different values for $\Delta x, \Delta y$, and Δz—a larger step size can be used along the axis of the waveguide than used to discretize the transverse space.

IV. Conclusion

This technique is relatively simple and yields accurate scattering parameters for discontinuous microstrip structures. Single- and multiple-step discontinuities can be investigated. This approach has a much smaller computational domain than would be necessary for a FDTD analysis of the entire physical structure. The reduced computational domain yields a corresponding reduction in computation time. Since this is a temporal technique, the scattering parameters can be obtained over a fairly broad spectrum using a single simulation.

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Scattering by a Thick Off-Centered Circular Iris in Circular Waveguide

Zhongxiang Shen and Robert H. MacPhie

Abstract—**A formally exact solution is described for the problems of scattering at a junction between two circular waveguides with their axes offset and at a thick off-centered iris in circular waveguide. The analysis method uses Graf's addition theorem for cylindrical functions and the conservation of complex power technique (CCPT). Sample numerical results are presented and compared with available data in the literature.**

I. Introduction

Waveguide iris coupling has found many applications in microwave engineering. Circular irises can be used as matching elements in microwave circuits or in waveguide filters. The problem of a circular iris in circular waveguide or the related step junction of two circular waveguides has been considered by many authors. Marcuvitz [1] gave the equivalent shunt susceptance for the TE_{11} mode excitation of small apertures. English [2] studied the mode conversion at a symmetric step-discontinuity in circular waveguide. Scharstein and Adams [3], [4] treated the problems of a TE_{11} mode circular waveguide with thin and thick circular irises. Carin *et al.* [5] investigated dielectric matched windows in circular waveguide. Most of the previously published works, however, have been limited to circular irises concentric with the axis of the circular waveguide. Although a simple expression of the equivalent shunt susceptance is available in [1] for the off-centered iris in circular waveguide, the expression is roughly approximate, and limited to the case of small aperture and of zero-thickness.

This paper gives a formally exact solution for the problem of a thick off-centered circular iris in circular waveguide. The conservation of complex power technique (CCPT), which has been used to obtain theoretically exact solutions with numerically convergent results to the problem of scattering at certain waveguide junctions [6], [7], and [8], and Graf's addition theorem for Bessel functions [9] are employed to obtain an analytical solution for the scattering matrix of a junction between two circular waveguides with their axes offset. The generalized scattering matrix technique [10] is then applied to deduce the scattering parameters of the off-centered iris in circular waveguide. Numerical results are presented and compared with those obtained by the approximate formula given in the *Waveguide Handbook* [1].

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II. Formulation

Fig. 1 shows the structure of a circular waveguide of radius a_2 loaded with an off-centered circular iris of radius a_1. (d, θ) are the polar coordinates of the center of the small circular waveguide in the coordinate system with its origin at the center of the larger waveguide (as illustrated in Fig. 1). Since the CCPT was well documented in [6], [7], and [8], only a summary of the formulation will be given here. The four scattering submatrices of the junction between guides 1 and 2 shown in Fig. 1 have the following form

$$\boldsymbol{S}_{11} = (\boldsymbol{Y}_1 + \boldsymbol{M}^T\boldsymbol{Y}_2\boldsymbol{M})^{-1}(\boldsymbol{Y}_1 - \boldsymbol{M}^T\boldsymbol{Y}_2\boldsymbol{M}) \tag{1}$$

$$\boldsymbol{S}_{21} = \boldsymbol{M}(\boldsymbol{S}_{11} + \boldsymbol{I}) \tag{2}$$

$$\boldsymbol{S}_{12} = \boldsymbol{Y}_1^{-1}S_{21}^T\boldsymbol{Y}_2 \tag{3}$$

$$\boldsymbol{S}_{22} = \boldsymbol{M}\boldsymbol{S}_{12} - \boldsymbol{I} \tag{4}$$

with $\boldsymbol{Y}_i$, for $i = 1$ and 2, is the modal admittance matrix for the ith waveguide [8], the superscript T denotes the transpose operation, and $\boldsymbol{M}$ is the E-field mode-matching matrix whose (nm, ki)th element is given by

$$M_{nm,\,ki} = \int_{S_1} (\vec{e}_{2,\,nm} \cdot \vec{e}_{1,\,ki})\, ds \tag{5}$$

where $\vec{e}_{i,\,nm}$ $(i = 1, 2)$ being the normalized transverse component of the nmth mode electric field of guide i, which has the form as follows

$$\vec{e}_{i,\,nm} = \begin{matrix} \hat{z} \times \nabla_t \psi^h_{i,\,nm}, & \text{for TE modes} \\ \nabla_t \psi^e_{i,\,nm}, & \text{for TM modes} \end{matrix} \tag{6}$$

where

$$\psi^h_{i,\,nm} = N^h_{nm} J_n\left(\frac{\rho_i x'_{nm}}{a_i}\right)\begin{pmatrix}\sin(n\varphi_i)\\ \cos(n\varphi_i)\end{pmatrix} \tag{7}$$

$$\psi^e_{i,\,nm} = N^e_{nm} J_n\left(\frac{\rho_i x_{nm}}{a_i}\right)\begin{pmatrix}\cos(n\varphi_i)\\ \sin(n\varphi_i)\end{pmatrix} \tag{8}$$

with

$$N^h_{nm} = \sqrt{\frac{\epsilon_n}{\pi}}\frac{1}{\sqrt{(x'_{nm})^2 - n^2}J_n(x'_{nm})} \tag{9}$$

$$N^e_{nm} = \sqrt{\frac{\epsilon_n}{\pi}}\frac{1}{x_{nm}J_{n+1}(x_{nm})} \tag{10}$$

being normalization constants, $\epsilon_n = 1$ for $n = 0$, and 2 for $n > 0$, and x'_{nm} and x_{nm} are, respectively, the mth zeros of $J'_n(x)$ and $J_n(x)$.

Since the integration in (5) is over the cross section of the small waveguide S_1, we must employ a coordinate transformation between